Analysis of Dynamic Properties of Macroeconomic Model

Katarína Makovínyiová

Abstract

In this paper a three dimensional model, which describes the development of output, interest rate and money supply in a closed economy is analysed.

We find the sufficient conditions for the existence of equilibrium, its stability and the existence of business cycles. Formulae for the calculation of bifurcation coefficients are derived for specific forms of functions in the model. Also the stability of business cycles is examined.

Key words

dynamical model, equilibrium, stability, bifurcation, cycles

AMS subject classifications: 37Gxx.

Introduction

In [4] there was analyzed the following macroeconomic model with pure money financing (also called Schinasi's Business Cycle Model, see [5])

$$\begin{aligned} \dot{Y} &= \alpha[I(Y,R) + G - S(Y^D) - T(Y)] \\ \dot{R} &= \beta[L(Y,R) - L_S] \\ \dot{L}_S &= G - T(Y)\,, \end{aligned} \tag{1}$$

where Y - output, R - interest rate, L_S - money supply, I - investments, S - savings, G - constant government expenditures, T - tax collections, L - money demand, α , β - positive parameters, t - time and

$$Y^D = Y - T(Y)\,, \quad \dot{Y} = \frac{dY}{dt}\,, \quad \dot{R} = \frac{dR}{dt}\,, \quad \dot{L}_S = \frac{dL_S}{dt}\,.$$

In this paper we shall treat the function S as the function of two variables Y^D and R. Thus we realize the function S in the model (1) as $S = S(Y^D, R)$ instead of $S = S(Y^D)$.

The economic properties of the functions in (1) are expressed by the following partial derivatives:

$$\frac{\partial I(Y,R)}{\partial Y} > 0, \ \frac{\partial I(Y,R)}{\partial R} < 0, \ \frac{\partial S(Y^D,R)}{\partial Y} > 0, \ \frac{\partial S(Y^D,R)}{\partial R} > 0,$$

$$\frac{\partial T(Y)}{\partial Y} > 0, \ \frac{\partial L(Y,R)}{\partial Y} > 0, \ \frac{\partial L(Y,R)}{\partial R} < 0. \tag{2}$$

The functions I , S , T , L in (1) were considered in general form assuming only that they had continuous derivatives up to the fifth order in the domain $M = \{(Y,R) : Y > 0,\ R > 0\}$.

In [4] there was proved the existence of limit cycles in the model (1). These results were generalized in [7]. In this paper we shall investigate the stability of cycles. The contribution starts with some complementary statements and their proofs.

Analysis of the model

In the following theorem a sufficient condition for the existence of an isolated equilibrium E^* of the model (1) is presented.

Theorem 1. *Suppose that* $T(0) < G$ *and* $I(Y^{\circ},0) > S(Y^{\circ},0)$*, where the value* Y° *is determined by the relation* $T(Y^{\circ}) = G$*. Then there exists the unique equilibrium* $E^* = (Y^*, R^*, L_S{}^*)$*,* $Y^* > 0,$ $R^* > 0$*,* $L_S{}^* > 0$ *of the model (1).*

Proof. From the third equation it follows that at equilibrium $E^* = (Y^*, R^*, L_S{}^*)$ the equation $G - T(Y^*) = 0$ is satisfied. The function T is increasing and if $T(0) < G$ the equation $G - T(Y^*) = 0$ determines the unique positive equilibrium value Y^*, $Y^* > 0$. Taking this into account we get from the first equation that $I(Y^*, R^*) = S(Y^*, R^*)$ must be satisfied. This equation has a unique positive solution R^* in the power of the facts that $I_R < 0,$ $S_R > 0$ and functions I, S satisfy the condition $I(Y^{\circ},0) > S(Y^{\circ},0)$, where the value Y° is the equilibrium value Y^*. The positive equilibrium value $L_S{}^*$ is determined by the equation $L(Y^*, R^*) - L_S{}^* = 0.$□

Henceforward we shall assume in the whole paper that the functions T, L are linear and the functions I, S are nonlinear of the partial Kaldor's form ([2]). We suggest the following forms of the functions:

$$T(Y) = t_0 + t_1 Y,$$
$$L(Y, R) = l_0 + l_1 Y - l_2 R$$
$$I(Y, R) = i_0 + i_1 \sqrt{Y} - i_2 R,$$
$$S(Y^D, R) = s_0 + s_1 \left(Y^D\right)^2 + s_2 R, \quad Y^D = Y - T(Y),$$

where t_1, l_1, l_2, i_1, i_2, s_1, s_2 are positive constants. After these functions' specification the model (1) takes the form

$$\begin{aligned} \dot{Y} &= \alpha\left[k_3 \sqrt{Y} + k_2 Y^2 + k_1 Y - (i_2 + s_2) R + k_0\right] \\ \dot{R} &= \beta[l_1 Y - l_2 R + l_0 - L_S] \\ \dot{L}_S &= -t_1 Y + G - t_0, \end{aligned} \tag{3}$$

where
$$k_3 = i_1,$$
$$k_2 = -s_1 (1 - t_1)^2,$$
$$k_1 = 2 t_0 s_1 (1 - t_1) - t_1,$$
$$k_0 = i_0 - s_0 - t_0 - s_1 t_0{}^2 + G.$$

Utilizing Theorem 1 we shall find a sufficient condition for the existence of an isolated equilibrium of (3).

Lemma 1. *If the inequalities* $G - t_0 > 0$*,* $i_0 + i_1 \sqrt{\dfrac{G - t_0}{t_1}} > s_0 + s_1 \left(\dfrac{G - t_0}{t_1} - G\right)^2$ *are satisfied then the model (3) has an isolated equilibrium point* $E^* = (Y^*, R^*, L_S{}^*)$*,* $Y^* > 0,$ $R^* > 0$*,* $L_S{}^* > 0$.

Consider an isolated equilibrium $E^* = (Y^*, R^*, L_S{}^*)$ of (3). After the transformation

$$Y_1 = Y - Y^*, \; R_1 = R - R^*, \; L_{S1} = L_S - L_S{}^*,$$

the equilibrium E^* shifts into the origin $E_1{}^* = (Y_1{}^* = 0, R_1{}^* = 0, L_{S1}{}^* = 0)$ and the model (3) takes the form

$$\dot{Y} = \alpha\left[k_3\sqrt{Y_1 + Y^*} + k_2(Y_1 + Y^*)^2 + k_1(Y_1 + Y^*) - (i_2 + s_2)(R_1 + R^*) + k_0\right]$$
$$\dot{R} = \beta[l_1(Y_1 + Y^*) - l_2(R_1 + R^*) + l_0 - L_{S1} - L_S{}^*] \tag{4}$$
$$\dot{L}_S = -t_1(Y_1 + Y^*) + G - t_0.$$

Performing Taylor expansion of the functions on the right-hand side of this system at the equilibrium $E_1{}^* = (0,0,0)$ we get the model

$$\dot{Y}_1 = \alpha[(B_{Y1} - t_1)Y_1 - (i_2 + s_2)R_1] + \widetilde{Y}_1(Y_1, R_1, \alpha)$$
$$\dot{R}_1 = \beta[l_1 Y_1 - l_2 R_1 - L_{S1}] \tag{5}$$
$$\dot{L}_{S1} = -t_1 Y_1,$$

where $B_{Y1} = \dfrac{\partial B}{\partial Y_1}(Y_1{}^*, R_1{}^*) = \dfrac{i_1}{2\sqrt{Y^*}} - 2s_1(1 - t_1)\left(Y^*(1 - t_1) - t_0\right)$ and the function $\widetilde{Y}_1$ contains corresponding nonlinear terms.

The linear approximation matrix of (5) is

$$\mathrm{A}(\alpha, \beta) = \begin{pmatrix} \alpha(B_{Y1} - t_1) & -\alpha(i_2 + s_2) & 0 \\ \beta l_1 & -\beta l_2 & -\beta \\ -t_1 & 0 & 0 \end{pmatrix}. \tag{6}$$

The characteristic equation of $\mathrm{A}(\alpha, \beta)$ is

$$\lambda^3 + a_1(\alpha, \beta)\lambda^2 + a_2(\alpha, \beta)\lambda + a_3(\alpha, \beta) = 0, \tag{7}$$

where $a_1 = \alpha(t_1 - B_{Y1}) + \beta l_2,$

$a_2 = \alpha\beta((i_2 + s_2)l_1 + (t_1 - B_{Y1})l_2),$

$a_3 = \alpha\beta t_1(i_2 + s_2).$

Stability of E_1^* is ensured by the Routh-Hurwitz conditions:

$$a_1 > 0,\ a_3 > 0,\ a_1 a_2 - a_3 > 0.$$

If
$$t_1 - B_{Y1} > 0 \quad \Leftrightarrow \quad t_1 + 2s_1(t_1 - 1)\left(t_0 + (t_1 - 1)Y^*\right) - \frac{i_1}{2\sqrt{Y^*}} > 0, \tag{8}$$

the conditions $a_1 > 0$, $a_3 > 0$ are satisfied. The inequality $a_1 a_2 - a_3 > 0$ is satisfied if

$$\alpha > \frac{t_1(i_2 + s_2)}{(t_1 - B_{Y1})\left(l_1(i_2 + s_2) + l_2(t_1 - B_{Y1})\right)} - \frac{\beta l_2}{t_1 - B_{Y1}}. \tag{9}$$

Theorem 2. Take parameter β at any positive level. If conditions (8) and (9) hold, all eigenvalues of matrix A have negative real parts. Hence the equilibrium E_1^* of the model (5) is asymptotically stable and the same result holds for the equilibrium point E^* of the model (3).

Now we are dealing with the question of the existence of cycles in the model (3). Accordingly we need to find such values of parameters $\alpha, \beta,$ at which the equation (7) has a pair of purely imaginary eigenvalues and the rest one is real and negative. We shall call such values of parameters α, β as critical values of the equation (7). We denote the critical values as α_0, β_0. Mentioned types of eigenvalues are ensured by the Liu's conditions ([3]):

$$a_1 > 0,\ a_3 > 0,\ a_1 a_2 - a_3 = 0. \tag{10}$$

The first two inequalities are satisfied with condition (8). The condition $a_1 a_2 - a_3 = 0$ implies that

$$\big(l_1(i_2+s_2)+l_2(t_1-B_{Y1})\big)\big(\beta l_2+\alpha(t_1-B_{Y1})\big)-t_1(i_2+s_2)=0.$$

As we are interested in positive critical values, take $\beta\in\left(0,\dfrac{t_1(i_2+s_2)}{l_1l_2(i_2+s_2)+l_2^{\,2}(t_1-B_{Y1})}\right)$ and denote it as critical value β_0. Then critical value α_0 is a positive number

$$\alpha_0=\frac{t_1(i_2+s_2)-\beta_0 l_2\big(l_1(i_2+s_2)+l_2(t_1-B_{Y1})\big)}{(t_1-B_{Y1})\big(l_1(i_2+s_2)+l_2(t_1-B_{Y1})\big)}.$$

Thus for given specific value of β_0 it is always possible to find corresponding value of α_0 so that a pair (α_0,β_0) is a critical pair of the equation (7) and of the model (5).

Lemma 2. Let the condition (8) holds and let (α_0,β_0) be a critical pair of the model (5). Then at every $\alpha>\alpha_0$ the equilibrium E_1^* is asymptotically stable and at every $\alpha<\alpha_0$ the equilibrium E_1^* is unstable.

Proof. We can rewrite the condition $a_1a_2-a_3=0$ in the form $\beta=\dfrac{t_1(i_2+s_2)}{l_1l_2(i_2+s_2)+l_2^{\,2}(t_1-B_{Y1})}-\dfrac{(t_1-B_{Y1})}{l_2}\alpha=q+k\alpha$, which is the linear function with a variable α. Let assume the condition (8) holds. Then constant q is positive and slope k is negative, so the function is decreasing and passing through the first quadrant (see Fig. 1).

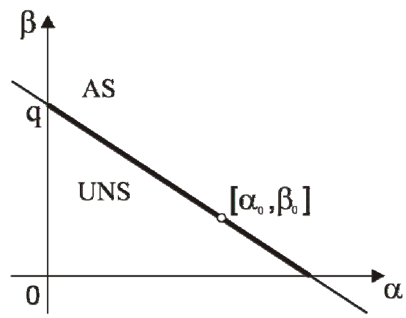

Figure 1.

Treat the parameter $\beta=\beta_0\in(0,q)$ as fixed. For $\alpha>\alpha_0$ we get $a_1a_2-a_3>0$ and for $\alpha<\alpha_0$ is $a_1a_2-a_3<0$. Using the Routh-Hurwitz stability conditions we get the assertion of the Lemma 2. □

To gain the bifurcation equation of the model (5) it is suitable to transform (5) to its partial normal form on invariant surface. After the shift of α_0 into the origin by relation $\alpha_1=\alpha-\alpha_0$ the model (5) takes the form

$$\begin{aligned}\dot{Y}_1&=\alpha_0[(B_{Y1}-t_1)Y_1-(i_2+s_2)R_1]+(B_{Y1}-t_1)Y_1\alpha_1-(i_2+s_2)R_1\alpha_1+\sum_{k=2}^{4}\alpha_0a_k\,Y_1^{\,k}+\sum_{k=2}^{4}\alpha_1a_k\,Y_1^{\,k}+O(|Y_1|^5)\\ \dot{R}_1&=\beta_0[l_1Y_1-l_2R_1-L_{S1}]\\ \dot{L}_{S1}&=-t_1Y_1,\end{aligned}\tag{11}$$

where $a_2=-s_1(1-t_1)^2-\dfrac{i_1}{8\sqrt{(Y^*)^3}}$, $a_3=\dfrac{i_1}{16\sqrt{(Y^*)^5}}$, $a_4=-\dfrac{5i_1}{128\sqrt{(Y^*)^7}}$.

Consider the matrix M which transfers the matrix $\mathsf{A}(\alpha_0, \beta_0)$ into its Jordan form J. Then the transformation $x = M.y$, $x = (Y_1, R_1, L_{S1})^T$, $y = (Y_2, R_2, L_{S2})^T$ takes the model (11) into the model

$$\begin{aligned}\dot{Y}_2 &= \lambda_1 Y_2 + F_1(Y_2, R_2, L_{S2}, \alpha_1)\\ \dot{R}_2 &= \lambda_2 R_2 + F_2(Y_2, R_2, L_{S2}, \alpha_1)\\ \dot{L}_{S2} &= \lambda_3 L_{S2} + F_3(Y_2, R_2, L_{S2}, \alpha_1),\end{aligned} \tag{12}$$

where $R_2 = \overline{Y_2}, F_2 = \overline{F_1}$, and F_3 is real (the symbol " - " means complex conjugate expression in the whole article).

Theorem 3. There exists a polynomial transformation

$$\begin{aligned}Y_2 &= Y_3 + h_1(Y_3, R_3, \alpha_1)\\ R_2 &= R_3 + h_2(Y_3, R_3, \alpha_1)\\ L_{S2} &= L_{S3} + h_3(Y_3, R_3, \alpha_1),\end{aligned} \tag{13}$$

where $h_j(Y_3, R_3, \alpha_1)$, $j = 1,2,3,$ are nonlinear polynomials with constant coefficients of the kind

$h_j(Y_3, R_3, \alpha_1) = \sum_{m_1, m_2, m_3} \nu^{(m_1, m_2, m_3)} Y_3^{m_1} R_3^{m_2} \alpha_1^{m_3}$, $j = 1,2,3,$ $h_2 = \overline{h_1}$, with the property $h_j(\sqrt{\alpha_1} Y_3, \sqrt{\alpha_1} R_3, \alpha_1) = \sum_{m_1, m_2, m_3} \nu^{(m_1, m_2, m_3)} (\sqrt{\alpha_1})^k Y_3^{m_1} R_3^{m_2}, k \le 4,$ which transforms the model (12) into its partial normal form on invariant surface

$$\begin{aligned}\dot{Y}_3 &= \lambda_1 Y_3 + \delta_1 Y_3 \alpha_1 + \delta_2 Y_3^2 R_3 + U^0(Y_3, R_3, L_{S3}, \alpha_1) + U^*(Y_3, R_3, L_{S3}, \alpha_1)\\ \dot{R}_3 &= \lambda_2 R_3 + \overline{\delta}_1 R_3 \alpha_1 + \overline{\delta}_2 Y_3 R_3^2 + \overline{U}^0(Y_3, R_3, L_{S3}, \alpha_1) + \overline{U}^*(Y_3, R_3, L_{S3}, \alpha_1)\\ \dot{L}_{S3} &= \lambda_3 L_{S3} + V^0(Y_3, R_3, L_{S3}, \alpha_1) + V^*(Y_3, R_3, L_{S3}, \alpha_1),\end{aligned} \tag{14}$$

where $U^0(Y_3, R_3, 0, \alpha_1) = V^0(Y_3, R_3, 0, \alpha_1) = 0$ and
$U^*(\sqrt{\alpha_1} Y_3, \sqrt{\alpha_1} R_3, \sqrt{\alpha_1} L_{S3}, \alpha_1) = V^*(\sqrt{\alpha_1} Y_3, \sqrt{\alpha_1} R_3, \sqrt{\alpha_1} L_{S3}, \alpha_1) = O(\sqrt{\alpha_1})^5.$

The resonant terms δ_1 and δ_2 in the model (14) are determined by the formulae

$$\delta_1 = \frac{\partial^2 F_1}{\partial Y_2 \partial \alpha_1},$$

$$\delta_2 = \frac{1}{\lambda_2}\frac{\partial^2 F_1}{\partial Y_2^2}\frac{\partial^2 F_1}{\partial Y_2 \partial R_2} + \frac{1}{6\lambda_1}\frac{\partial^2 F_2}{\partial Y_2^2}\frac{\partial^2 F_1}{\partial R_2^2} + \frac{1}{\lambda_1}\frac{\partial^2 F_1}{\partial Y_2 \partial R_2}\frac{\partial^2 F_2}{\partial Y_2 \partial R_2} +$$

$$+\frac{1}{2\lambda_1}\frac{\partial^2 F_1}{\partial Y_2^2}\frac{\partial^2 F_1}{\partial Y_2 \partial R_2} + \frac{(-1)}{\lambda_3}\frac{\partial^2 F_1}{\partial Y_2 \partial R_2}\frac{\partial^2 F_3}{\partial Y_2 \partial R_2} + \frac{1}{2(2\lambda_1 - \lambda_3)}\frac{\partial^2 F_1}{\partial R_2 \partial L_{S2}}\frac{\partial^2 F_3}{\partial Y_2^2} + \frac{1}{2}\frac{\partial^3 F_1}{\partial Y_2^2 \partial R_2},$$

where all derivatives in δ_1 and δ_2 are calculated at E_1^* and $\alpha_1 = 0$.

Proof. Differentiating (13) in the power of (12) and (14) we get the equations for the determination of the individual terms of the polynomials h_j, $j = 1,2,3,$ and the resonant terms

δ_1, δ_2 by standard "step by step" procedure. As the whole process of this procedure is rather long we are omitting it. □

The model (14) takes in polar coordinates $Y_3 = re^{i\varphi}$, $R_3 = re^{-i\varphi}$ the form

$$\begin{aligned}
\dot{r} &= r\left(ar^2 + b\alpha_1\right) + \widetilde{U}^0(r,\varphi,L_{S3},\alpha_1) + \widetilde{U}^*(r,\varphi,L_{S3},\alpha_1) \\
\dot{\varphi} &= \omega + c\alpha_1 + dr^2 + \frac{1}{r}\left[\Phi^0(r,\varphi,L_{S3},\alpha_1) + \Phi^*(r,\varphi,L_{S3},\alpha_1)\right] \\
\dot{L}_{S3} &= \lambda_3 L_{S3} + \widetilde{V}_1^0(r,\varphi,L_{S3},\alpha_1) + \widetilde{V}_1^*(r,\varphi,L_{S3},\alpha_1),
\end{aligned} \tag{15}$$

where $a = \operatorname{Re}\delta_2$, $b = \operatorname{Re}\delta_1$.

The behaviour of solutions of the model (15) around its equilibrium for small parameters α_1 depends on the signs of the constants a,b. It is known that to every constant solution of the bifurcation equation $ar^2 + b\alpha_1 = 0$ a periodic solution of (15) corresponds (see [1]).

The following lemma gives a sufficient condition for the negativeness of the coefficient b.

Lemma 3. *Let the assumptions of Lemma 2 be satisfied. Let in addition* $t_1 + l_1\lambda_3(\alpha_0,\beta_0) < 0$ *holds. Then the value of the coefficient* b *is negative.*

Proof. Denote the eigenvalues of (6) as $\lambda_1 = \tau(\alpha,\beta) + i\omega(\alpha,\beta)$, $\lambda_2 = \tau(\alpha,\beta) - i\omega(\alpha,\beta)$, $\lambda_3 = \lambda_3(\alpha,\beta)$. After substituting these roots into the equation (7) we get

$$\left(\lambda - (\tau + i\omega)\right)\left(\lambda - (\tau - i\omega)\right)(\lambda - \lambda_3) = 0,$$

what gives the equation

$$\lambda^3 + b_1(\alpha,\beta)\lambda^2 + b_2(\alpha,\beta)\lambda + b_3(\alpha,\beta) = 0, \tag{16}$$

where

$$\begin{aligned}
b_1 &= -(2\tau + \lambda_3), \\
b_2 &= 2\tau\lambda_3 + \tau^2 + \omega^2, \\
b_3 &= -(\tau^2 + \omega^2)\lambda_3.
\end{aligned}$$

Comparing the coefficients in (7) with the ones in (16) we gain the equations for $\tau(\alpha,\beta)$, $\omega(\alpha,\beta)$, $\lambda_3(\alpha,\beta)$:

$$\begin{aligned}
\alpha(t_1 - B_{Y1}) + \beta l_2 &= -(2\tau + \lambda_3), \\
\alpha\beta((i_2 + s_2)l_1 + (t_1 - B_{Y1})l_2) &= 2\tau\lambda_3 + \tau^2 + \omega^2, \\
\alpha\beta t_1(i_2 + s_2) &= -(\tau^2 + \omega^2)\lambda_3.
\end{aligned} \tag{17}$$

Let (α_0,β_0) be a critical pair of the model (5). Fix β_0 and consider a neighbourhood $O(\alpha_0) = (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$, $\varepsilon > 0$. It is known that the constant b from the bifurcation equation $ar^2 + b\alpha_1 = 0$ equals to the derivation of $d\tau(\alpha)/d\alpha$ at the critical values α_0, β_0 (see [6]). Hence put β_0 from the critical pair (α_0,β_0) into (17). Derivation of (17) with respect to α at the critical value α_0 taking into account that $\tau(\alpha_0) = 0$ gives

$$
\begin{aligned}
2\frac{d\tau}{d\alpha} \qquad\qquad &+ \frac{d\lambda_3}{d\alpha} = B_{Y1} - t_1,\\
2\lambda_3\frac{d\tau}{d\alpha} + 2\omega\frac{d\omega}{d\alpha} + \qquad &= \beta_0\left(l_1(i_2+s_2)+l_2(t_1-B_{Y1})\right),\\
2\omega\lambda_3\frac{d\omega}{d\alpha} &+ \omega^2\frac{d\lambda_3}{d\alpha} = -\beta_0 t_1(i_2+s_2).
\end{aligned}
$$

By Cramer rule we get from this system

$$\frac{d\tau(\alpha_0)}{d\alpha} = \frac{G}{H},$$

where $G = \beta_0(i_2+s_2)(t_1+l_1\lambda_3)+(t_1-B_{Y1})(l_2\beta_0\lambda_3-\omega^2)$ and $H = 2(\lambda_3{}^2+\omega^2)$. Denominator H is always positive. If $t_1+l_1\lambda_3<0$ is satisfied, G is negative. On the base of performed considerations we can formulate Lemma 3. □

Analysing the model (15) and taking into account all transformations which have been done to get (15) we can formulate on the base of Poincaré-Andronov-Hopf bifurcation theorem (see [6]) the following statement.

Theorem 4. *Let the assumptions of Lemma 2 be satisfied. Let in addition* $t_1+l_1\lambda_3(\alpha_0,\beta_0)<0$ *holds. Then:*

1. *If* $a>0$ *the equilibrium* E_1^* *of the model (5) is unstable also at the critical pair* (α_0,β_0) *and to every* $\alpha>\alpha_0$ *there exists an unstable limit cycle.*
2. *If* $a<0$ *the equilibrium* E_1^* *of the model (5) is asymptotically stable also at the critical pair* (α_0,β_0) *and to every* $\alpha<\alpha_0$ *there exists a stable limit cycle.*

Conclusion

In Theorem 1 a sufficient condition for the existence of an isolated equilibrium of the model (1) is presented. Lemma 1 gives a sufficient condition for the existence of equilibrium of the model (1) with respect to specific forms of functions occurring in the model.

Theorem 2 uses the Routh-Hurwitz conditions to state the equilibrium stability of (5) and (3). Lemma 2 solves the question of equilibrium stability in the case of one free parameter on the specified model (5). It was shown that to an arbitrary parameter $\beta=\beta_0$ it is always possible to find a corresponding value of the parameter α, $\alpha=\alpha_0$, such that we are able to answer the question about equilibrium stability in the interval $(\alpha_0-\varepsilon,\alpha_0+\varepsilon)$, $\varepsilon>0$.

Theorem 3 gives formulae for the calculation of bifurcation coefficients. The question about the existence of business cycles and their stability in the model (5) is completely answered by Theorem 4.

References:

[1] Bibikov, Yu. N.: Multi-frequency non-linear oscillations and their bifurcations, The Publishing House of the Saint Petersburg University, Saint Petersburg, 1991 (in Russian).

[2] Kaldor, N.: A Model of the Trade Cycle, Economic Journal 50, 1940, pp. 78-92.

[3] Liu, W. M.: Criterion of Hopf Bifurcations without using Eigenvalues, Journal of Mathematical Analysis and Applications 182, 1994, pp. 250-256.

[4] Sasakura, K.: On the Dynamic Behavior of Schinasi's Business Cycle Model, Journal of Macroeconomics, Vol. 16 (1994), No. 1, pp. 423-444.

[5] Schinasi, G. J.: Fluctuations in a Dynamic, Intermediate-Run IS-LM Model: Applications of the Poincare-Bendixson Theorem, Journal of Economic Theory 28, 1982.

[6] Wiggins, S.: Introduction to Applied Nonlinear Dynamical Systems and Chaos, Springer-Verlag, Berlin, 1990.

[7] Zimka, R.: Existence of Hopf bifurcation in IS-LM model depending on more-dimensional parameter, Proceedings of "International Scientific Conference on Mathematics", Herľany, Slovak Republic, 1999, pp. 176-182.

Katarína Makovínyiová
Technical University in Zvolen
T. G. Masaryka 2117/24
960 53 Zvolen
Slovak Republic
kmakovin@vsld.tuzvo.sk